

# Representations of Spacetime Alternatives and Their Classical Limits

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Different quantum mechanical operators can correspond to the same classical quantity. Hermitian operators differing only by operator ordering of the canonical coordinates and momenta at one moment of time are the most familiar example. Classical spacetime alternatives that extend over time can also be represented by different quantum operators. For example, operators representing a particular value of the time average of a dynamical variable can be constructed in two ways: First, as the projection onto the value of the time averaged Heisenberg picture operator for the dynamical variable. Second, as the class operator defined by a sum over those histories of the dynamical variable that have the specified time-averaged value. We show both by explicit example and general argument that the predictions of these different representations agree in the classical limit and that sets of histories represented by them decohere in that limit.

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## I. INTRODUCTION

Usual quantum mechanics predicts the probabilities of alternatives specified at a moment in time and histories of such alternatives specified at a sequence of times. A single particle moving in one dimension provides a familiar example. The probability  $p(\Delta)$  that the particle's position  $x$  lies in a range  $\Delta$  at a time  $t$  is given in the Heisenberg picture by

$$p(\Delta) = \|P_{\Delta}^x(t)|\psi\rangle\|^2. \quad (1.1)$$

Here,  $P_{\Delta}^x(t)$  is the projection onto the range  $\Delta$  of the eigenvalues of the operator  $x$  at a time  $t$ , and  $|\psi\rangle$  is the particle's state.

A given classical alternative, for example  $2x^3p^2$  (where  $p$  is the momentum), can correspond to several distinct quantum mechanical alternatives, e.g.,  $x^3p^2 + p^2x^3$ ,  $xp^2x^2 + x^2p^2x$ ,  $xpxpx + xpxpx$ . These differ by operator ordering. Probabilities of suitably coarse grained ranges of such alternatives approximately agree for states representing classical situations. This paper explores different operator representations of a more general type of alternative for a single non-relativistic particle — spacetime alternatives extended over time.

Classical alternatives are not restricted to definite moments of time. Consider the continuous average of position over the range of times between 0 and  $T$ , specifically

$$\bar{x} \equiv \frac{1}{T} \int_0^T x(t) dt. \quad (1.2)$$

This is a simple example of a spacetime alternative that is not at one time, but rather is extended over time.

The general notion of spacetime alternatives for a single particle moving in one dimension is a partition of the set of paths of the particle into an exhaustive set of mutually exclusive classes. For example, we could partition the paths into classes defined by whether the values of a functional, such as Eq. (1.2), fall into one or another of an exhaustive set of ranges  $\{\Delta_{\alpha}\}$ ,  $\alpha = 1, 2, \dots$ . We could partition the paths by whether or not they cross a given spacetime region between two times, etc. Alternatives at a single moment of time are just a special case of this more general class.

Spacetime alternatives in the sense of field averages occur routinely in field theory [1]. Spacetime alternatives may permit more realistic descriptions of measurements. No realistic measurement occurs exactly at one moment in time. Finally, spacetime alternatives may be essential for a quantum theory of gravity, where there is no definite notion of spacetime geometry to supply meaning to “at a moment of time” [2].

How are spacetime alternatives represented in quantum mechanics? The consideration of spacetime alternatives in quantum mechanics has a long history. The discussions in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] are just some of the many examples that could be cited. In the context of non-relativistic quantum mechanics, a comprehensive treatment can be given in the sum-over-histories quantum mechanics of closed systems [8, 9]. The essential feature used in this paper is the following: If the paths  $x(t)$  between  $t = 0$  and  $t = T$  are partitioned into an exhaustive set of mutually exclusive classes  $c_{\alpha}$ ,  $\alpha = 1, 2, \dots$ , then the operators  $\hat{C}_{\alpha}$  representing the individual alternatives in this set of histories are defined in the Schrödinger picture<sup>1</sup> by sums over the histories in  $c_{\alpha}$ .

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<sup>1</sup> We use a hat to distinguish the class operator defined by the path integral (1.3) in the Schrödinger picture from the sums of chains of projections  $C_{\alpha}$  useful in the Heisenberg picture. They are related by  $\hat{C}_{\alpha} = \exp(-iHT/\hbar)C_{\alpha}$ . Both operators give the same

Specifically these sums have the form

$$\langle x'' | \hat{C}_\alpha | x' \rangle \equiv \int_{c_\alpha} \delta x \exp \left( \frac{i}{\hbar} S[x(t)] \right). \quad (1.3)$$

Here,  $S[x(t)]$  is the action functional. The sum is over all paths  $x(t)$  which start at  $x'$  at  $t = 0$ , end at  $x''$  at  $t = T$ , and are in the class  $c_\alpha$ . As an example, consider the set of histories defined by whether the time average  $\bar{x}$  in Eq.(1.2) lies in one or another of an exhaustive set of exclusive ranges  $\{\Delta_\alpha\}$ ,  $\alpha = 1, 2, \dots$ . The path integral that defines the class operator  $\hat{C}_\Delta$  for the alternative that  $\bar{x}$  lies in the particular range  $\Delta$  in the set  $\{\Delta_\alpha\}$  is over all paths with the above starting and ending values for which  $\bar{x}$  has a value in  $\Delta$ . Its probability is

$$p_{\text{soh}}(\Delta) = \|\hat{C}_\Delta |\psi\rangle\|^2. \quad (1.4)$$

The set of probabilities defined by Eq.(1.4) for all  $\Delta$  in the set  $\{\Delta_\alpha\}$  are generally not consistent with the rules of probability theory unless a decoherence condition is satisfied [8, 9]. For instance, the probability for  $\bar{x}$  to lie in a range  $\Delta$  and the probability to lie in the complementary range  $\bar{\Delta}$  would not generally sum to one. An example of a condition which ensures that such relations are satisfied is the medium decoherence condition.

$$\langle \psi | \hat{C}_\Delta^\dagger \hat{C}_{\Delta'} | \psi \rangle \approx 0, \quad \Delta \neq \Delta'. \quad (1.5)$$

When this is satisfied for all set pairs  $\Delta \neq \Delta'$  drawn from the  $\{\Delta_\alpha\}$  the set of histories is said to *decohere*.

The time average  $\bar{x}$  also defines a Hermitian operator in the Heisenberg picture. It is therefore also natural to think of its probability as being given by

$$p_{\text{proj}}(\Delta) = \|P_\Delta |\psi\rangle\|^2 \quad (1.6)$$

where  $P_\Delta$  is the projection operator onto the eigenstates of the operator  $\bar{x}$  defined by Eq.(1.2). Decoherence is not an issue for the alternatives defined by the set projections onto the ranges  $\{\Delta_\alpha\}$ . The analog of the decoherence condition Eq.(1.5) is automatically satisfied because projections onto different ranges are exactly orthogonal:

$$P_\Delta P_{\Delta'} = 0, \quad \Delta \neq \Delta'. \quad (1.7)$$

Eq. (1.6) is the rule usually given for the probabilities of measurements of fields averaged over spacetime regions [1]. There are questions as to what such measurements might mean [11], when their outcomes are accurately predicted by Eq. (1.6), and how to assign probabilities to sequences of such measurements[10]. This paper leaves such interesting issues aside and instead concentrates on analyzing the mathematical difference between Eq. (1.4) and Eq. (1.6) in simple models, asking whether their predictions coincide in the classical limit, and determining whether or not the alternatives  $\hat{C}_\alpha$  decohere in that limit.

At the level of operators, it is convenient to compare  $\hat{C}_\Delta$ , not with  $P_\Delta$  directly, but rather with the combination

$$\hat{P}_\Delta \equiv e^{-iHT/\hbar} P_\Delta \quad (1.8)$$

which is not a projection but gives the same probabilities as Eq. (1.6) and satisfies [cf. Eq.(1.7)]

$$\hat{P}_\Delta^\dagger \hat{P}_{\Delta'} = 0, \quad \Delta \neq \Delta'. \quad (1.9)$$

This comparison is useful because  $\hat{P}_\Delta$  and  $\hat{C}_\Delta$  coincide in the limit of large widths of  $\Delta$ .

In Sections IV and V we calculate the action of  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  on simple wave functions for two models where they can be evaluated explicitly—a non-relativistic free particle and a non-relativistic harmonic oscillator. We show that the two operators are quantitatively nearly the same when acting on states describing classical situations. Further we show that they coincide in a formal  $\hbar \rightarrow 0$  limit. Section VI gives a general argument why this is the case, and Section VII shows how sets of such alternatives decohere in the  $\hbar \rightarrow 0$  limit. We conclude that  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  can be regarded as distinct quantum mechanical representations of the same classical alternative.

## II. FORMALISM

In this section we will briefly outline how to construct the quantum operators  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  defined in Eqs. (1.3) and (1.8) that represent the spacetime alternative  $\bar{x} \in \Delta$  defined in Eq. (1.2).

The coarse grained history of interest  $c_\Delta$  is the class of histories  $x(t)$  defined as follows

$$c_\Delta = \left\{ x(t) \left| \bar{x} \equiv \frac{1}{T} \int_0^T dt x(t) \in \Delta \right. \right\} \quad (2.1)$$

where  $\Delta$  is the subset of the real line of width  $\delta$  centered on  $x_c$ :

$$\Delta = \{x | x_c - \delta/2 \leq x \leq x_c + \delta/2\}. \quad (2.2)$$

That is, the class  $c_\Delta$  consists of all paths that start at  $x'$  at  $t = 0$  and end at  $x''$  at  $t = T$ , such that the average value of the path  $\bar{x}$  is in the range  $\Delta$ . The operator  $\hat{C}_\Delta$  is defined by the path integral in Eq. (1.3) over paths in the class  $c_\Delta$ . The operator  $\hat{P}_\Delta$  is defined by:

$$\begin{aligned} \langle x'' | \hat{P}_\Delta | x' \rangle &= \langle x'' | e^{-iHT/\hbar} P_\Delta | x' \rangle \\ &= \int_\Delta d\bar{x} \langle x'' | e^{-iHT/\hbar} | \bar{x} \rangle \langle \bar{x} | x' \rangle. \end{aligned} \quad (2.3)$$

How do these two operators differ? Clearly  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  are not equal.  $\hat{P}_\Delta$  is proportional to a single projection operator, while  $\hat{C}_\Delta$  is proportional to an infinite

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probabilities in (1.4).

product of projection operators which in general is not a projection operator. However, for large  $\delta$ ,  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  approach each other. To see this, let us examine how  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  are constructed in more detail.

The eigenstates of  $\bar{x}$  form a complete set with delta function normalization:

$$\int_{-\infty}^{\infty} d\bar{x} |\bar{x}\rangle \langle \bar{x}| = I, \quad \langle \bar{x} | \bar{x}' \rangle = \delta(\bar{x} - \bar{x}'). \quad (2.4)$$

The projection,  $P_\Delta$  is defined by:

$$P_\Delta = \int_{\Delta} d\bar{x} |\bar{x}\rangle \langle \bar{x}|. \quad (2.5)$$

As  $\delta$  increases, the integral in Eq. (2.5) is over a larger and larger portion of the real line defined by Eq. (2.2). Thus, from Eq. (2.4), as  $\delta$  approaches infinity,  $P_\Delta$  approaches unity, and

$$\lim_{\delta \rightarrow \infty} \langle x'' | e^{-iHT/\hbar} P_\Delta | x' \rangle = \langle x'' | e^{-iHT/\hbar} | x' \rangle \equiv K(x'', T; x', 0) \quad (2.6)$$

where  $K(x'', T; x', 0)$  is the propagator from  $t = 0$  to  $t = T$ .

For  $\langle x'' | \hat{C}_\Delta | x' \rangle$  we are integrating over all paths between  $x'$  at  $t = 0$  and  $x''$  at  $t = T$ , such that  $\bar{x} \in \Delta$ . Thus, as  $\delta$  approaches infinity, the class of paths being integrated over becomes less and less restricted, and

$$\lim_{\delta \rightarrow \infty} \int_{c_\Delta} \delta x e^{iS[x(t)]/\hbar} = \int_u \delta x e^{iS[x(t)]/\hbar} = K(x'', T; x', 0) \quad (2.7)$$

where the unrestricted functional integral on the right hand side is over *all* paths between  $x'$  at  $t = 0$  and  $x''$  at  $t = T$ . Therefore, in the limit of large  $\delta$ , we find

$$\langle x'' | \hat{P}_\Delta | \psi \rangle \approx \langle x'' | \hat{C}_\Delta | \psi \rangle. \quad (2.8)$$

for suitable initial states  $|\psi\rangle$ . The scale of  $\delta$  above which this approximate equality holds is set by the spatial extent of the wave function  $\psi(x, t)$  over the time interval  $t \in (0, T)$ .

In subsequent sections we evaluate and compare  $\langle x'' | \hat{P}_\Delta | \psi \rangle$  and  $\langle x'' | \hat{C}_\Delta | \psi \rangle$  explicitly using Gaussian initial wave functions  $\psi(x)$  for two simple systems: the one-dimensional, non-relativistic free particle, and the one-dimensional, non-relativistic harmonic oscillator.

### III. GENERAL POTENTIAL

Consider a one-dimensional quantum system with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \quad (3.1)$$

In Section IV we will set  $V(x) = 0$ , and in Section V we will set  $V(x) = \frac{1}{2}m\omega^2 x^2$ , but here we discuss results that do not depend on these specific forms of  $H$ .

In the Heisenberg picture, an operator  $O$  evolves in time according to the equation of motion

$$i\hbar \frac{dO}{dt} = [O, H]. \quad (3.2)$$

For  $p$  and  $x$ , this gives the following coupled evolution equations

$$\dot{x}(t) = \frac{p}{m}, \quad \dot{p}(t) = -\frac{dV(x)}{dx}. \quad (3.3)$$

With a suitable operator ordering prescription, these equations can be solved for  $x(t)$  and  $p(t)$  in terms of  $p(0) \equiv p_0$  and  $x(0) \equiv x_0$ . We then construct  $\bar{x}$  from the Heisenberg picture operator  $x(t)$ :

$$\bar{x} = \frac{1}{T} \int_0^T dt x(t). \quad (3.4)$$

The eigenstates  $|\bar{x}\rangle$  of  $\bar{x}$  form a complete set with delta function normalization, Eq. (2.4). One can construct an operator representation of the alternative  $\bar{x} \in \Delta$  by projecting onto the eigenvalues of  $\bar{x}$ , as in Eq. (2.5). For the time period starting at  $t = 0$  and ending at  $t = T$ , this operator has matrix elements in the position basis given by

$$\begin{aligned} \langle x'' | \hat{P}_\Delta | x' \rangle &= \langle x'' | e^{-iHT/\hbar} P_\Delta | x' \rangle \\ &= \int_{\Delta} d\bar{x} \int_{-\infty}^{\infty} dy \langle x'' | e^{-iHT/\hbar} | y \rangle \langle y | \bar{x} \rangle \langle \bar{x} | x' \rangle. \end{aligned} \quad (3.5)$$

Calculating the class operator  $\hat{C}_\Delta$  for the spacetime alternative  $\bar{x} \in \Delta$  is slightly less straightforward. We begin with  $\hat{C}_\Delta$  defined in (1.3). The functional integral can be rewritten by introducing the top-hat function

$$e_\Delta(z) = \begin{cases} 1, & z \in \Delta \\ 0, & z \notin \Delta \end{cases} \quad (3.6)$$

and its Fourier transform

$$e_\Delta(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikz} \tilde{e}_\Delta(k). \quad (3.7)$$

Thus we may write Eq. (1.3) as

$$\begin{aligned} \langle x'' | \hat{C}_\Delta | x' \rangle &\equiv \int_{c_\Delta} \delta x e^{iS[x(t)]/\hbar} \\ &= \int_u \delta x e_\Delta(\bar{x}[x(t)]) e^{iS[x(t)]/\hbar} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{e}_\Delta(k) \int_u \delta x e^{iS[x(t)]/\hbar + ik\bar{x}} \end{aligned} \quad (3.8)$$

where the unrestricted path integral is now over all paths from  $x'$  at  $t = 0$  to  $x''$  at  $t = T$  and  $\bar{x}[x(t)]$  is the time averaging functional defined by Eq. (3.4). Eq. (3.8) allows us to define an effective action

$$S_{\text{eff}}[x(t)] = S[x(t)] + \hbar k \bar{x} \quad (3.9)$$

and an effective Lagrangian

$$L_{\text{eff}} = L + \frac{\hbar}{T} kx(t). \quad (3.10)$$

We can take this calculation a step further by evaluating  $\tilde{e}_\Delta(k)$ . Note that  $e_\Delta(z) = \int_\Delta dy \delta(z-y)$ . Thus

$$\tilde{e}_\Delta(k) = \frac{2}{\sqrt{2\pi}} \frac{1}{k} e^{-ikx_c} \sin\left(\frac{k\delta}{2}\right)$$

and the class operator is thus given by

$$\langle x'' | \hat{C}_\Delta | x' \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dk}{k} e^{-ikx_c} \sin\left(\frac{k\delta}{2}\right) \int_u \delta x e^{iS_{\text{eff}}[x(t)]/\hbar}. \quad (3.11)$$

We are now prepared to examine these expressions for specific quantum systems. We discuss the free particle in Section IV and the harmonic oscillator in Section V.

#### IV. THE FREE PARTICLE

##### A. $\hat{P}_\Delta$ and $\hat{C}_\Delta$

We begin by examining the free particle:  $L = \frac{1}{2}m\dot{x}^2$ . For this system the Heisenberg equations can be solved:  $x(t) = x_0 + (p_0/m)t$  and  $p(t) = p_0$ . Thus,

$$\bar{x} = x_0 + \frac{p_0}{2m}T. \quad (4.1)$$

In the position basis, the eigenstates of  $\bar{x}$  are solutions of the equation

$$x \langle x | \bar{x} \rangle + \frac{T}{2m} \frac{\hbar}{i} \frac{d}{dx} \langle x | \bar{x} \rangle = \bar{x} \langle x | \bar{x} \rangle. \quad (4.2)$$

Solving this equation and imposing the delta function normalization, Eq. (2.4), gives:

$$\langle x | \bar{x} \rangle = \sqrt{\frac{m}{\pi \hbar T}} \exp\left[\frac{i}{\hbar} \frac{2m}{T} \left(\bar{x}x - \frac{x^2}{2}\right)\right]. \quad (4.3)$$

We substitute this expression into Eq.(3.5) along with the expression for the propagator of the free particle:

$$\begin{aligned} \langle x'' | e^{-iHT/\hbar} | x' \rangle &\equiv K(x'', T; x', 0) \\ &= \sqrt{\frac{m}{2\pi i \hbar T}} \exp\left[\frac{i}{\hbar} \frac{m}{2T} (x'' - x')^2\right]. \end{aligned} \quad (4.4)$$

Integrating Eq. (3.5) over  $\bar{x}$  and  $y$  gives:

$$\langle x'' | \hat{P}_\Delta | x' \rangle = K(x'', T; x', 0) E_\Delta\left(\frac{x'' + x'}{2}, \lambda\right). \quad (4.5)$$

where  $\Delta$  is the range  $[x_c - \delta/2, x_c + \delta/2]$  and the length  $\lambda$  is defined by

$$\lambda \equiv \left(\frac{\hbar T}{2m}\right)^{1/2} = 2.3 \times 10^{-14} \left(\frac{1\text{g}}{m}\right)^{1/2} \left(\frac{T}{1\text{s}}\right) \text{cm}. \quad (4.6)$$

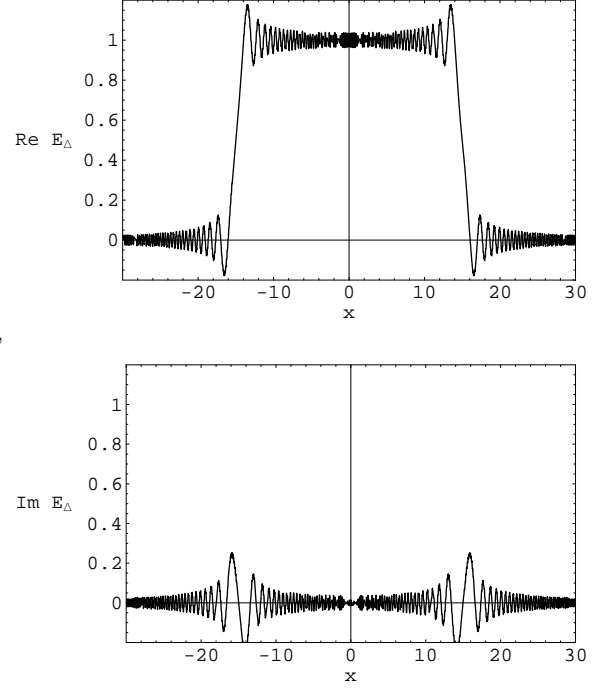


FIG. 1: The real and imaginary parts of the function  $E_\Delta(z, \ell)$  defined by (4.7). These graphs are for  $\delta/\lambda = 15$ . Already at this value the real part is a reasonable approximation to a top hat function  $e_\Delta(z/\ell)$  and the imaginary part is small. As  $\hbar$  approaches zero, the value of  $\lambda$  becomes very small [cf Eq. (4.6)], the approximation of the real part to a top-hat function in both the expression Eq. (4.5) for  $\langle x'' | \hat{P}_\Delta | x' \rangle$  and Eq. (4.9) for  $\langle x'' | \hat{C}_\Delta | x' \rangle$  becomes better and better and the imaginary part becomes increasingly negligible.

Here,  $E_\Delta(z, \ell)$  is the function on the range  $\Delta = [a, b]$  defined by

$$E_\Delta(z, \ell) \equiv \frac{1}{2} \left[ \text{erf}\left(\frac{z-a}{\sqrt{i}\ell}\right) - \text{erf}\left(\frac{z-b}{\sqrt{i}\ell}\right) \right] \quad (4.7)$$

Figure 1 illustrates the function  $E_\Delta(z, \ell)$ . Its important property for the subsequent discussion is that it approaches the top-hat function  $e_\Delta$  as the dimensionless ratio  $\delta/\ell$  becomes large. Figure 1 shows that this is not a bad approximation when that ratio is only 15.

Our next task is to calculate  $\langle x'' | \hat{C}_\Delta | x' \rangle$  according to Eq. (3.8). For the free particle, the effective Lagrangian is Eq. (3.10). This is a well known system with a well known propagator [14]:

$$\begin{aligned} \int_u \delta x e^{iS_{\text{eff}}[x(t)]/\hbar} &= \left(\frac{m}{2\pi i \hbar T}\right)^{1/2} \exp\left\{\frac{i}{\hbar} \left[\frac{m(x'' - x')^2}{2T}\right.\right. \\ &\quad + \frac{1}{2} \left(\frac{\hbar k}{T}\right) T(x'' + x') \\ &\quad \left.\left. - \left(\frac{\hbar k}{T}\right)^2 \frac{T^3}{24m}\right]\right\}. \end{aligned} \quad (4.8)$$

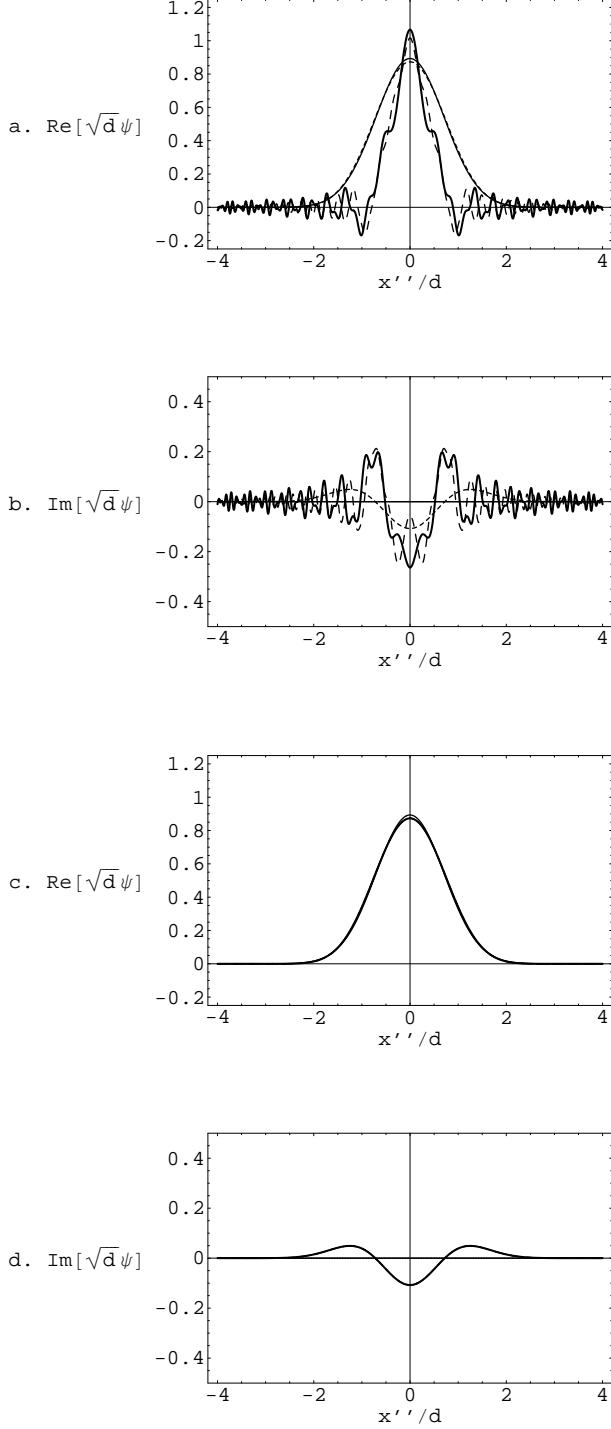


FIG. 2: Graph of  $\sqrt{d}\psi$  for the free particle where  $\psi$  equals  $\psi(x'', 0)$  (solid line),  $\psi(x'', T)$  (small dashed line),  $\langle x'' | \hat{C}_\Delta | \psi \rangle$  (large dashed line), and  $\langle x'' | \hat{P}_\Delta | \psi \rangle$  (thick line). For a. and b.,  $\delta = d$ . For c. and d.,  $\delta = 10d$ . In all graphs,  $T = t_{\text{spread}}/4$ .

Substituting this expression into Eq. (3.11) and evaluating the integral over  $k$  yields

$$\langle x'' | \hat{C}_\Delta | x' \rangle = K(x'', T; x', 0) E_\Delta \left( \frac{x'' + x'}{2}, \frac{\lambda}{\sqrt{3}} \right) \quad (4.9)$$

where  $\lambda$  was defined in Eq. (4.6).

Clearly the two operators Eq. (4.5) and Eq. (4.9) have a similar mathematical structure. Indeed they differ only by the factor of  $\sqrt{3}$  in the argument of the function  $E_\Delta$ . However, they are not equal. Examining how  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  act on a known wave function will give us an idea of how the two operators differ. To that end we examine  $\langle x'' | \hat{C}_\Delta | \psi \rangle$  and  $\langle x'' | \hat{P}_\Delta | \psi \rangle$  where  $|\psi\rangle$  has a Gaussian wave function of width  $d$ .

$$\psi(x) = \left( \frac{2}{\pi d^2} \right)^{1/4} \exp \left( -\frac{x^2}{d^2} \right) \quad (4.10)$$

With this Gaussian initial wave packet, we might expect classical behavior for spacetime alternatives when two conditions are satisfied: (1) The coarse-graining of position  $\delta$  is much larger than the quantum uncertainty in position  $d$  specified by the wave packet. (2) The time  $T$  over which the alternative is defined is much smaller than the wave packet spreading time of order the combination

$$t_{\text{spread}} \equiv d^2 m / 2\hbar. \quad (4.11)$$

These two conditions can be neatly expressed in terms of the length  $\lambda$  introduced in Eq.(4.6) as

$$\frac{\delta}{\lambda} \gg \frac{d}{\lambda} \gg 1. \quad (4.12)$$

This is also the condition under which  $E_\Delta$  is well approximated by a top-hat function as discussed earlier. Eq.(4.6) shows that typical “macroscopic” coarse grainings satisfy these criteria easily.

Fig. 2 shows the real and imaginary parts of  $\langle x'' | \hat{C}_\Delta | \psi \rangle$  and  $\langle x'' | \hat{P}_\Delta | \psi \rangle$  for two cases, each with  $T = t_{\text{spread}}/4$ , but with  $\delta = d$  in one case and  $\delta = 10d$  in the other. As expected, the two quantities are significantly different in the first case when Eq.(4.12) is not satisfied, and very close in the second case when it is satisfied. However, as discussed in Section II, for  $\delta \gg d$ , there is surprisingly good agreement between the time evolved wavefunction  $\psi(x'', T)$ ,  $\langle x'' | \hat{C}_\Delta | \psi \rangle$ , and  $\langle x'' | \hat{P}_\Delta | \psi \rangle$ , as can be seen in Fig. (2c-d.).

## B. The Classical Limit

We now compare  $\langle x'' | \hat{C}_\Delta | x' \rangle$  and  $\langle x'' | \hat{P}_\Delta | x' \rangle$  in the classical limit,  $\hbar \rightarrow 0$ . Here we are dealing with a familiar situation in quantum mechanics. We have two distinct quantum operators,  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$ , each of which represents the same classical spacetime alternative. It

is worth exploring whether the two operators represent exactly the same classical alternative.

For both  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$ , we use the method of stationary phase to evaluate the expressions in the  $\hbar \rightarrow 0$  limit. Let us begin with  $\hat{C}_\Delta$ . Recall,

$$\langle x'' | \hat{C}_\Delta | x' \rangle = \int_u \delta x e_\Delta(\bar{x}[x(t)]) e^{iS[x(t)]/\hbar}. \quad (4.13)$$

We may introduce a change in path variable

$$x(t) = x_{\text{cl}}(t) + y(t) \quad (4.14)$$

where  $x_{\text{cl}}(t)$  is the classical path. That is,  $x_{\text{cl}}(t)$  satisfies the classical equations of motion for the action  $S$  with  $x_{\text{cl}}(0) = x'$  and  $x_{\text{cl}}(T) = x''$ . Substituting Eq. (4.14) into Eq. (4.13) above gives:

$$\begin{aligned} \langle x'' | \hat{C}_\Delta | x' \rangle &\approx \int_u \delta y e_\Delta(\bar{x}[x_{\text{cl}}(t) + y(t)]) e^{iS[x_{\text{cl}}(t)]/\hbar} \\ &\times \exp\left(\frac{i}{\hbar} \int_0^T dt \frac{1}{2} m \dot{y}^2\right). \end{aligned} \quad (4.15)$$

where we have used  $L = m\dot{x}^2/2$  and assumed there is a unique classical path. In the  $\hbar \rightarrow 0$  limit, the integral is dominated by the value of the integrand at the saddle point  $y = 0$ . Thus, the top hat function can be pulled out of the path integral. Noting that the remaining path integral is the propagator for the free particle with  $y' = y'' = 0$  we find,

$$\langle x'' | \hat{C}_\Delta | x' \rangle \sim e_\Delta(\bar{x}_{\text{cl}}) K(x'', T; x', 0) \quad (4.16)$$

where  $\bar{x}_{\text{cl}}$  is the time average (1.2) of the classical path  $x_{\text{cl}}(t)$ . The classical equation of motion is  $m\ddot{x}_{\text{cl}} = 0$ . Solving this equation with the above boundary conditions yields a unique classical path:

$$x_{\text{cl}}(t) = (x'' - x') \frac{t}{T} + x'. \quad (4.17)$$

Thus,

$$\bar{x}_{\text{cl}} = \frac{x'' + x'}{2}. \quad (4.18)$$

Substituting this into Eq. (4.16) gives the following asymptotic form of the class operator for the free particle as  $\hbar \rightarrow 0$ :

$$\langle x'' | \hat{C}_\Delta | x' \rangle \sim e_\Delta\left(\frac{x' + x''}{2}\right) K(x'', T; x', 0). \quad (4.19)$$

We evaluate the expression (3.5) for  $\langle x'' | \hat{P}_\Delta | x' \rangle$  in the  $\hbar \rightarrow 0$  limit by careful application of the stationary phase approximation [15]

$$\int_a^b dt e^{izh(t)} g(t) \sim \frac{e^{izh(t_0)} \sqrt{2\pi}}{\sqrt{z} \sqrt{\pm h''(t_0)}} e^{\pm \pi i/4} g(t_0) \quad (4.20)$$

as  $z \rightarrow \infty$  where  $h'(t_0) = 0$  for  $t_0 \in (a, b)$ , and  $h''(t_0) \neq 0$ . We use the plus signs if  $h''(t_0) > 0$ , and the minus signs if  $h''(t_0) < 0$ . Substituting Eq. (4.3) and Eq. (4.4) into Eq. (3.5), and using Eq. (4.20) to evaluate the integrals over  $y$  and  $\bar{x}$  in the  $\hbar \rightarrow 0$  limit yields

$$\langle x'' | \hat{P}_\Delta | x' \rangle \sim e_\Delta\left(\frac{x' + x''}{2}\right) K(x'', T; x', 0). \quad (4.21)$$

The same result could be obtained from the observation made earlier that  $E_\Delta(z, \ell)$  approaches  $e_\Delta(z)$  as  $\delta/\ell$  becomes large.

A comparison of (4.19) and (4.21) shows that the operators  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  coincide in the classical limit. Their classical predictions will be the same.

## V. THE HARMONIC OSCILLATOR

The calculation of  $\hat{P}_\Delta$  and  $\hat{C}_\Delta$  for the harmonic oscillator parallels that of the free particle. We begin with the harmonic oscillator Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2. \quad (5.1)$$

Solving the Heisenberg equations for  $x$  and  $p$  and calculating  $\bar{x}$  yields

$$\bar{x} = \frac{\sin \omega T}{\omega T} x_0 + \frac{1 - \cos \omega T}{(\omega T)^2} \frac{p_0}{m} T. \quad (5.2)$$

Again we find the eigenstates of  $\bar{x}$ , and impose delta function normalization. In the position basis,  $|\bar{x}\rangle$  is given by

$$\langle x | \bar{x} \rangle = \sqrt{\frac{m}{\pi \hbar T}} \frac{\omega T/2}{|\sin(\omega T/2)|} \exp\left[\frac{i}{\hbar} \frac{m}{T} \frac{(\omega T)^2}{1 - \cos(\omega T)} \left(\bar{x}x - \frac{1}{2} \frac{\sin \omega T}{\omega T} x^2\right)\right]. \quad (5.3)$$

Substituting this expression into Eq. (3.5) along with the propagator for the harmonic oscillator (cf.[14]), and in-

tegrating over  $y$  and  $\bar{x}$  yields:

$$\langle x'' | \hat{P}_\Delta | x' \rangle = K(x'', T; x', 0) E_\Delta(\bar{x}_{\text{cl}}, \lambda_P), \quad (5.4)$$

where  $\bar{x}_{\text{cl}}$  is the average (1.2) of the position along the classical path given by

$$\bar{x}_{\text{cl}} = \left( \frac{1 - \cos \omega T}{\omega T \sin \omega T} \right) (x' + x''). \quad (5.5)$$

and  $\lambda_P$  is defined by

$$\lambda_P \equiv \left[ \frac{2\hbar T}{m} \frac{\omega T}{\sin \omega T} \left( \frac{1 - \cos \omega T}{(\omega T)^2} \right)^2 \right]^{1/2} \quad (5.6)$$

In order to calculate  $\langle x'' | \hat{C}_\Delta | x' \rangle$  for the harmonic oscillator, we note that

$$L_{\text{eff}} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 + \frac{\hbar k}{T} x. \quad (5.7)$$

Again, this system has a known propagator [14]:

$$\int_u \delta x e^{iS_{\text{eff}}/\hbar} = \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega T}} e^{iS_{\text{eff,cl}}/\hbar} \quad (5.8)$$

where

$$\begin{aligned} S_{\text{eff,cl}} = & \frac{m\omega}{2 \sin \omega T} \left\{ (x''^2 + x'^2) \cos \omega T - 2x'x'' \right. \\ & + \frac{2}{m\omega} \frac{\hbar k}{T} \frac{1 - \cos \omega T}{\omega} (x' + x'') \\ & \left. - \frac{2}{m^2 \omega^2} \left( \frac{\hbar k}{T} \right)^2 \left[ \frac{1 - \cos \omega T}{\omega^2} - \frac{T}{2\omega} \sin \omega T \right] \right\}. \end{aligned} \quad (5.9)$$

Substituting Eq. (5.8) into Eq. (3.11), and evaluating the integral over  $k$  yields

$$\langle x'' | \hat{C}_\Delta | x' \rangle = K(x'', T; x', 0) E_\Delta(\bar{x}_{\text{cl}}, \lambda_C), \quad (5.10)$$

where

$$\lambda_C \equiv \left[ \frac{4\hbar T}{m} \frac{\omega T}{\sin \omega T} \left( \frac{1 - \cos \omega T}{(\omega T)^4} - \frac{1}{2} \frac{\sin \omega T}{(\omega T)^3} \right) \right]^{1/2} \quad (5.11)$$

and  $\bar{x}_{\text{cl}}$  is defined in Eq. (5.5).

Clearly  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  share a similar mathematical structure. However, they are not equal. Again, we examine how  $\hat{C}_\Delta$  and  $\hat{P}_\Delta$  act on a Gaussian wave function. The comparison parallels that of the free particle in Section IV. That is, for  $\delta \approx d$ ,  $\langle x'' | \hat{C}_\Delta | \psi \rangle$  and  $\langle x'' | \hat{P}_\Delta | \psi \rangle$  look rather different (Fig. (3a-b.)). However, in Fig. (3c-d.), we see that for  $\delta \gg d$ , there is surprisingly good agreement between  $\langle x'' | \hat{C}_\Delta | \psi \rangle$  and  $\langle x'' | \hat{P}_\Delta | \psi \rangle$ . Thus, we expect the two operators to produce similar results in classical situations.

We examine the classical limit. The integrals in the  $\hbar \rightarrow 0$  limit can be evaluated using the same techniques as in Section IV. As in the case of the free particle the matrix elements of the two operators are equal in the classical limit:

$$\langle x'' | \hat{C}_\Delta | x' \rangle \approx \langle x'' | \hat{P}_\Delta | x' \rangle \sim e_\Delta(\bar{x}_{\text{cl}}) K(x'', T; x', 0) \quad (5.12)$$

where  $\bar{x}_{\text{cl}}$  is given by Eq. (5.5). All the relations for the free particle discussed in the previous section can be recovered from the zero frequency limit of the relations for the harmonic oscillator in this section.

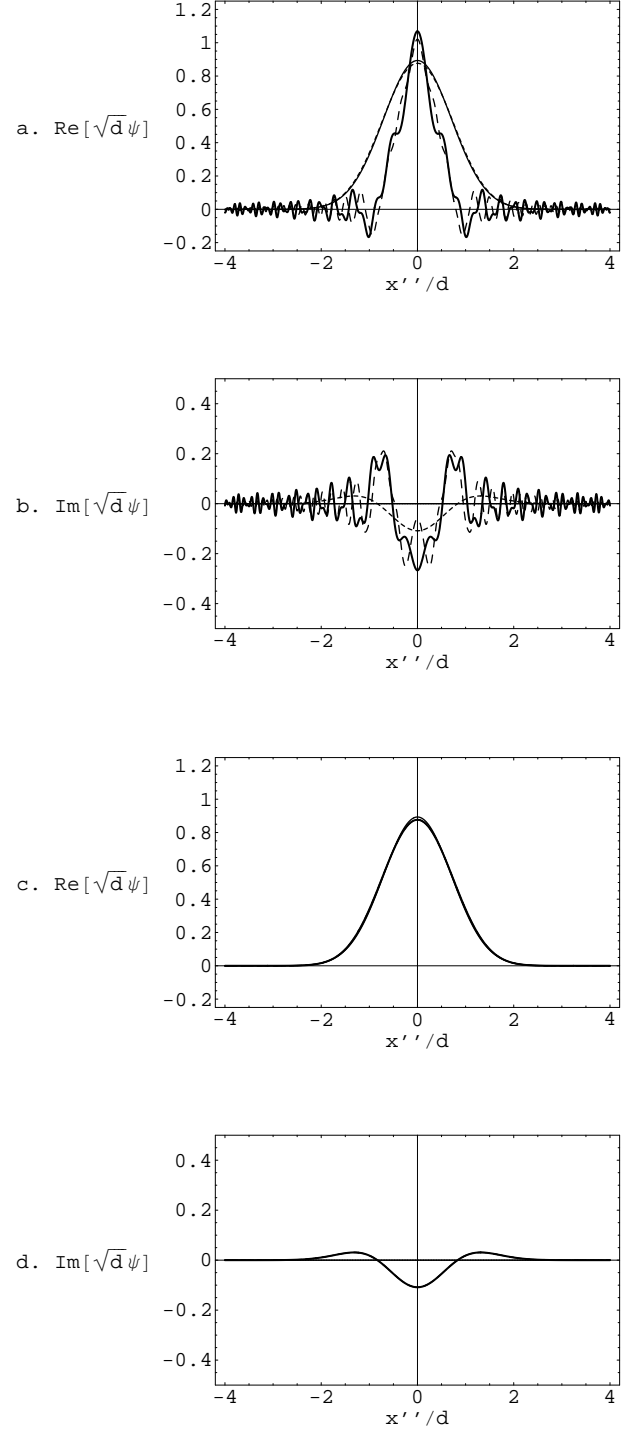


FIG. 3: Graph of  $\sqrt{d}\psi$  for the harmonic oscillator where  $\psi$  equals  $\psi(x'', 0)$  (solid line),  $\psi(x'', T)$  (small dashed line),  $\langle x'' | \hat{C}_\Delta | \psi \rangle$  (large dashed line), and  $\langle x'' | \hat{P}_\Delta | \psi \rangle$  (thick line). For a. and b.,  $\delta = d$ . For c. and d.,  $\delta = 10d$ . In all graphs,  $T = t_{\text{spread}}/4$  where  $t_{\text{spread}}$  is given by Eq. (4.11).

## VI. GENERAL ALTERNATIVES AND GENERAL POTENTIAL

The preceding sections demonstrate by example the agreement, in a formal  $\hbar \rightarrow 0$  classical limit, between probabilities calculated using class operators and Heisenberg picture projection operators for ranges of the spacetime alternative  $\bar{x}$ . This section sketches a demonstration of this classical agreement for general spacetime alternatives defined in terms of position and for general Hamiltonians of the form Eq. (3.1). Our results are essentially formal and not rigorous but suggest the underlying mechanisms of classical agreement.

We continue with a particle moving in one dimension described by a Hamiltonian of the form Eq. (3.1). The alternatives of interest are specified classically by a functional  $F_{\text{cl}}[x(t)]$  of the particle path  $x(t)$  between  $t = 0$  and  $t = T$ . For example, the spacetime alternative that the position time average  $\bar{x}$  lies in a range  $\Delta$  corresponds to the functional  $F_{\text{cl}}[x(t)] = e_{\Delta}(\bar{x}[x(t)])$  where  $\bar{x}[x(t)]$  is the functional defined by Eq. (3.4). These are the alternatives for which matrix elements of class operators can be defined by sums-over-histories of the form

$$\langle x'' | \hat{F}_{\text{soh}} | x' \rangle = \int_{x'} \delta x F_{\text{cl}}[x(t)] e^{iS[x(t)]/\hbar} \quad (6.1)$$

where the integration is over all paths  $x(t)$  which start at  $x'$  at  $t = 0$  and end at  $x''$  at  $t = T$ . The sum over histories Eq. (1.3) is of this form with  $F_{\text{cl}}[x(t)] = e_{\Delta}(\bar{x}[x(t)])$ .

The formal classical limit of path integrals like Eq. (6.1) is easily calculated. Assuming that  $F_{\text{cl}}$  does not itself depend on  $\hbar$ , the dominant contribution as  $\hbar \rightarrow 0$  comes from classical paths  $x_{\text{cl}}(t)$  between  $x'$  at  $t = 0$  and  $x''$  at  $t = T$  that extremize the action  $S[x(t)]$ . Assume for simplicity that there is only one such classical path  $x_{\text{cl}}(t, x'', x')$ . Because of the increasingly rapid varying phase as  $\hbar \rightarrow 0$ , the functional  $F_{\text{cl}}$  may be taken outside the integral and evaluated at this  $x_{\text{cl}}(t, x'', x')$ . Thus,

$$\langle x'' | \hat{F}_{\text{soh}} | x' \rangle = F_{\text{cl}}[x_{\text{cl}}(t, x'', x')] K(x'', T; x', 0) + \epsilon(x'', x'), \quad (6.2)$$

where  $K(x'', T; x', 0)$  is the propagator defined by the unrestricted, unweighted, sum-over-paths [cf. Eq. (2.7)]. Here and throughout, by  $\epsilon(x'', x')$  we mean *some function that goes to zero with  $\hbar$* , typically like  $\sqrt{\hbar}$ .

We now demonstrate the same classical limit for the matrix elements of

$$\hat{F}_{Hp} \equiv e^{-iHT/\hbar} F \quad (6.3)$$

where  $F$  is a Hermitian Heisenberg picture operator representing the classical functional  $F_{\text{cl}}[x(t)]$ . Different operator representations can be formed by first using the classical equations of motion to express  $F_{\text{cl}}$  as functions of the position  $x_0$  and momentum  $p_0$  at time  $t = 0$ . (Eq. (4.1) is an explicit example for the time average  $\bar{x}$ .) Then, the different Hermitian operator orderings for

this classical expression give different Heisenberg picture operators  $F$  representing the classical functional  $F_{\text{cl}}$ . We expect matrix elements of these to agree in the classical limit and they will.

Our construction of the classical limit of Eq. (6.3) relies on extrapolating results from microlocal analysis [16]. The central element of this formalism is the *symbol*  $\check{A}(X, P)$  of an operator  $A$  defined by

$$\check{A}(X, P) = \int d\xi e^{-iP\xi/\hbar} \langle X + \xi/2 | A | X - \xi/2 \rangle, \quad (6.4a)$$

with the inverse formula

$$\langle x'' | A | x' \rangle = \frac{1}{2\pi} \int dX \int dP \delta\left(X - \frac{x'' + x'}{2}\right) \times e^{iP(x'' - x')/\hbar} \check{A}(X, P). \quad (6.4b)$$

In the simple case  $A = |\psi\rangle\langle\psi|$ ,  $\check{A}(X, P)$  is the Wigner distribution for the state  $|\psi\rangle$ .

Two results from microlocal analysis concerning the classical limit will be important for us. The first concerns the symbol for the product of two operators. If  $C = AB$  then

$$\check{C}(X, P) = \check{A}(X, P) \check{B}(X, P) + \epsilon(X, P). \quad (6.5)$$

The second result concerns the time evolution of the operator  $A(t)$  given by the Heisenberg equations of motion Eq. (3.2). Express  $A(t)$  in terms of the position and momentum operators  $x_0$  and  $p_0$  at  $t = 0$ . The relation  $x(t) = x_0 + p_0 t/m$  for a free particle is a simple example. Construct the symbol  $\check{A}(t, X, P)$  using eigenstates of  $x_0$  as the basis in Eq. (6.4). Construct the symbol of the  $\check{H}(X, P)$  of the Hamiltonian in the same way. Then, to leading order in  $\hbar$ , the symbol for  $A$  obeys the classical equation of motion.

$$\frac{\partial \check{A}}{\partial t} = \{\check{A}, \check{H}\} + \epsilon(X, P) \quad (6.6)$$

where  $\{\cdot, \cdot\}$  is the Poisson bracket. Explicit forms of the corrections to Eq. (6.5) and Eq. (6.6) are given in [16]. We now employ these two results to calculate the classical limit of Eq. (6.3).

Matrix elements of Eq. (6.3) in the basis of eigenstates of  $x_0$  can be written

$$\langle x'' | \hat{F}_{Hp} | x' \rangle = \int dy \langle x'' | e^{-iHT/\hbar} | y \rangle \langle y | F | x' \rangle. \quad (6.7)$$

We concentrate first on the classical limit of  $\langle x'' | F | x' \rangle$  and later return to that for the whole expression Eq. (6.7).

We begin with the time evolution of the symbols for  $x$  and  $p$ . At  $t = 0$ ,  $\check{x}(0) = X$  and  $\check{p}(0) = P$ . Their time evolution to leading order in  $\hbar$  is determined by the two coupled classical equations (6.6) for  $\check{x}(t)$  and  $\check{p}(t)$  — Hamilton's equations of motion. We can therefore write

$$\check{x}(t, X, P) = x_{\text{cl}}(t, X, P) + \epsilon(t, X, P) \quad (6.8)$$



where  $x_{\text{cl}}(t, X, P)$  is the solution of the classical equation of motion with initial ( $t = 0$ ) position  $X$  and momentum  $P$ . For example, the symbol for time average  $\bar{x}$  is

$$\bar{x}(X, P) = \bar{x}_{\text{cl}}(X, P) + \epsilon(X, P) \quad (6.9)$$

where  $\bar{x}_{\text{cl}}$  is the classical time average. Some limitation on the size of the time  $T$  is likely to be needed to control the size of the corrections  $\epsilon$ .

More general functionals of  $x(t)$  can be treated as follows. First, solve the Heisenberg equations of motion and express  $x(t)$  in terms of operator products of  $x_0$ 's and  $p_0$ 's. Use Eq. (6.5) to show that the symbols of these products are the product of the symbols to leading order  $\hbar$ . Therefore, for any operator representative  $F$  of the classical functional  $F_{\text{cl}}[x(t)]$ , the classical limit of its symbol is

$$\tilde{F}(X, P) = F_{\text{cl}}[x_{\text{cl}}(t, X, P)] + \epsilon(X, P). \quad (6.10)$$

We now use this formula to calculate the classical limit of the matrix elements of  $\langle x'' | \hat{F}_{Hp} | x' \rangle$  defined using Eq. (6.4).

Using Eq. (6.4b), the expression Eq. (6.7) can be written

$$\begin{aligned} \langle x'' | \hat{F}_{Hp} | x' \rangle &= \frac{1}{2\pi} \int dy \int dP \langle x'' | e^{-iHT/\hbar} | y \rangle \\ &\times e^{iP(y-x')/\hbar} \tilde{F}\left(\frac{y+x'}{2}, P\right). \end{aligned} \quad (6.11)$$

The classical limit of the symbol  $\tilde{F}$  is given by Eq. (6.10). The classical limit of the propagator is given by the standard result

$$\begin{aligned} \langle x'' | e^{-iHT/\hbar} | y \rangle &\equiv K(x'', T; x', 0) \\ &= D^{-\frac{1}{2}}(x'', y) \exp[iS_{\text{cl}}(x'', y, T)/\hbar] + \epsilon(x'', y). \end{aligned} \quad (6.12)$$

Here,  $S_{\text{cl}}(x'', y, T)$  is the action of the classical path  $x_{\text{cl}}(t, x'', y)$  (assumed unique) that starts at  $y$  at  $t = 0$  and ends at  $x''$  at  $t = T$ . The slowly varying prefactor  $D$  is essentially the DeWitt-van Vleck determinant. Inserting Eq. (6.12) and Eq. (6.10) into Eq. (6.11), we find

$$\langle x'' | \hat{F}_{Hp} | x' \rangle = \frac{1}{2\pi} \int dy \int dP D^{-\frac{1}{2}}(x'', y) F_{\text{cl}}\left[x_{\text{cl}}\left(t, \frac{y+x'}{2}, P\right)\right] \exp\left\{\frac{i}{\hbar} [S_{\text{cl}}(x'', y, T) + P(y-x')]\right\} + \epsilon(x'', x'). \quad (6.13)$$

The rapid variation of the phase for small  $\hbar$  means that the double integral in Eq. (6.13) can be evaluated by the method of stationary phase. The dominant contribution comes from the  $(y, P)$  that extremize the exponential's phase. The conditions for this are

$$y = x' \quad , \quad P = -\frac{\partial S_{\text{cl}}}{\partial y}. \quad (6.14)$$

The second of these singles out the initial momentum  $P(x'', x', T)$  necessary for the classical path to arrive at  $x''$  a time  $T$  after it starts at  $x'$ . We thus find for the leading contribution

$$\begin{aligned} \langle x'' | \hat{F}_{Hp} | x' \rangle &= F_{\text{cl}}[x_{\text{cl}}(t, x'', x')] K(x'', T; x', 0) \\ &+ \epsilon(x'', x') \end{aligned} \quad (6.15)$$

where  $x_{\text{cl}}(t, x'', x')$  is the classical path from  $x'$  to  $x''$  in time  $T$ .

Comparison of Eq. (6.15) with Eq. (6.2) shows that the matrix elements of  $\langle x'' | \hat{F}_{Hp} | x' \rangle$  agree with  $\langle x'' | \hat{F}_{soh} | x' \rangle$  in leading classical order. This is the agreement demonstrated explicitly in Sections IV and V for the time average  $\bar{x}$  for the free particle and harmonic oscillator. The argument in this section is a general form of that.

## VII. DECOHERENCE OF SPACETIME ALTERNATIVES IN THE CLASSICAL LIMIT

Previous sections have shown how the representations of spacetime alternatives by sum-over-histories class operators and Heisenberg picture projection operators coincide in the classical limit. A general argument was given in Section VI and Sections IV and V provided specific examples. This section demonstrates the decoherence of exhaustive sets of histories coarse-grained by spacetime alternatives in the same classical limit. The essence of the argument is this: Decoherence is automatic and exact for sets of alternatives represented by projections [*cf.* Eq. (1.7)]. It is not necessarily automatic for sets represented by sum-over-histories class operators. However, in classical limit, where the class operators coincide with projections, sum-over-histories class operators must decohere also.

We continue to consider the histories of a particle moving in one dimension with a Hamiltonian Eq. (3.1) between time 0 and  $T$ . To keep the discussion manageable we restrict attention to coarse grainings by ranges of values of a functional of the paths  $f[x(t)]$ . The classical functionals  $F_{\text{cl}}[x(t)]$  considered in the previous section are a special case. The time average  $\bar{x}$  of Eq. (1.2) is an example. Specifically, we consider an exhaustive set of

exclusive ranges  $\{\Delta_\alpha\}$ ,  $\alpha = 1, 2, \dots$  and the set of alternative histories defined by the classes  $\{c_\alpha\}$  where  $f[x(t)]$  takes values in these ranges. The class operators for the set are defined by [cf. Eq. (1.3), Eq. (6.1)]

$$\langle x'' | \hat{C}_\Delta | x' \rangle \equiv \int_u \delta x e_\Delta(f[x(t)]) \exp\left(\frac{i}{\hbar} S[x(t)]\right). \quad (7.1)$$

where we denote by  $\Delta$  any of the intervals in the set  $\{\Delta_\alpha\}$ . This set of histories decoheres when the condition Eq. (1.5) is satisfied.

Projection operators onto distinct ranges are exactly orthogonal [cf. Eq. (1.7)].

$$P_\Delta P_{\Delta'} = \hat{P}_\Delta^\dagger \hat{P}_{\Delta'} = 0, \quad \Delta \neq \Delta'. \quad (7.2)$$

The decoherence condition Eq. (1.5) is therefore exact if the  $\hat{C}$ 's are replaced by  $P$ 's or  $\hat{P}$ 's. Since matrix elements of  $\hat{C}_\Delta$  coincide with those of  $\hat{P}_\Delta$  in the  $\hbar \rightarrow 0$  limit, we expect Eq. (1.5) to be satisfied in that limit as a consequence. We now demonstrate this explicitly.

Consider

$$\begin{aligned} \langle x | \hat{C}_\Delta^\dagger \hat{C}_{\Delta'} | x' \rangle &= \int dy \langle y | \hat{C}_\Delta | x \rangle^* \langle y | \hat{C}_{\Delta'} | x' \rangle \\ &= \int dy \int_u \delta x \int_u \delta x' e_\Delta(f[x(t)]) e_{\Delta'}(f[x'(t')]) \\ &\quad \times \exp\left\{\frac{i}{\hbar} (S[x'(t')] - S[x(t)])\right\}. \end{aligned} \quad (7.3)$$

The integrals are over paths  $x(t)$  and  $x'(t')$  that start at  $x$  and  $x'$  respectively at  $t = 0$  and  $t' = 0$  and both end at  $y$  at  $t = t' = T$ . In the  $\hbar \rightarrow 0$  limit the rapid oscillation of the exponential means that the integrals will be dominated by the stationary (classical) paths  $x_{\text{cl}}(t, y, x)$  and  $x_{\text{cl}}(t, y, x')$  connecting  $x$  and  $x'$  to  $y$ . (We suppose unique stationary paths for simplicity.) We assume that, as a consequence, the top-hat factors can be taken outside the path integrals and evaluated at the stationary paths. Thus in the  $\hbar \rightarrow 0$  limit we have

$$\langle x | \hat{C}_\Delta^\dagger \hat{C}_{\Delta'} | x' \rangle \sim \int dy e_\Delta(f[x_{\text{cl}}(t, y, x)]) e_{\Delta'}(f[x_{\text{cl}}(t, y, x')]) \exp\left\{\frac{i}{\hbar} (S_{\text{cl}}(y, x') - S_{\text{cl}}(y, x))\right\}, \quad (7.4)$$

where  $S_{\text{cl}}(y, x)$  is the action evaluated at the classical path with endpoints  $y$  and  $x$ .

When  $x = x'$  The exponent in Eq. (7.4) vanishes but so does the factor multiplying it when  $\Delta \neq \Delta'$ . The reason is that when  $x = x'$  the arguments of the top-hat functions are the same but their ranges are exclusive. When  $x \neq x'$  we assume that the derivative of the exponent with respect to  $y$  does not vanish identically. The

Riemann-Lebesgue lemma then shows that Eq. (7.4) vanishes as  $\hbar \rightarrow 0$ . Thus for all values of  $x$  and  $x'$  we confirm Eq. (1.5) and the decoherence of the set of histories in the classical limit.

The case of the histories of a free particle partitioned by ranges of the time average position  $\bar{x}$  that was worked out in detail in Section IV provides a ready example. Using Eq. (4.9) the matrix element in Eq. (7.3) is explicitly

$$\begin{aligned} \langle x | \hat{C}_\Delta^\dagger \hat{C}_{\Delta'} | x' \rangle &= \int dy \langle y | \hat{C}_\Delta | x \rangle^* \langle y | \hat{C}_{\Delta'} | x' \rangle = \left(\frac{m}{2\pi\hbar T}\right) \exp\left[\frac{i}{\hbar} \frac{m}{2T} (x'^2 - x^2)\right] \int dy \exp\left[\frac{i}{\hbar} \frac{m}{T} (x - x')y\right] \\ &\quad \times E_\Delta\left(\frac{y+x}{2}, \frac{\lambda}{\sqrt{3}}\right) E_{\Delta'}\left(\frac{y+x'}{2}, \frac{\lambda}{\sqrt{3}}\right). \end{aligned} \quad (7.5)$$

As discussed in Section IV,  $E_\Delta(z, \lambda/\sqrt{3}) \rightarrow e_\Delta(z)$  in the limit  $\hbar \rightarrow 0$  and  $\lambda$  becomes large. This allows  $E$ 's to be replaced by  $e$ 's in Eq. (7.5) evaluated at the classical path. For a classical path moving between  $x$  and  $y$  in a time  $T$ ,  $\bar{x} = (y + x)/2$ . This shows the argument of the  $e$ 's in Eq. (7.4) is just that of the argument of the  $E$ 's in Eq. (7.5). Thus the form Eq. (7.4) is recovered explicitly from Eq. (7.5).

## VIII. CONCLUSION

The lesson of both the special and general theories of relativity is that four-dimensional spacetime is the most general arena for physics on scales well above the Planck scale of quantum gravity. Correspondingly quantum theory is formulated most generally in four dimensional form in terms of sets of histories of spacetime alternatives that are extended in time, their decoherence, and their prob-

abilities. This paper has explored the classical ( $\hbar \rightarrow 0$ ) limit of quantum operators representing spacetime alternatives in the context of non-relativistic quantum theory. A given classical spacetime alternative may have many different representations in terms of quantum operators. We considered two kinds: (1) Class operators defined by sums over the classical histories of the alternative, and (2) projection operators on ranges of the Heisenberg picture operators. In Sections VI and VII we gave a general argument based on microlocal analysis for why the class operators for a set of exclusive alternatives decohere in the classical limit and why the predictions of both kinds of representations coincide in that limit. Our results were formal because we did not provide general estimates for the corrections to decoherence and the differences in probabilities for small  $\hbar$ . However, we analyzed these corrections explicitly for a particular alternative in particular tractable model systems in Sections III and IV. Specifically we considered the average of posi-

tion over a time interval  $\overline{\tau}$  as a simple spacetime alternative for the free particle and harmonic oscillator in one-dimension. We showed by explicit calculation how class operators and projections have differing matrix elements in general, but also how those coincide in the classical limit. These results show explicitly how class operators and corresponding projection operators can be different operator representations of the same classical spacetime alternative.

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